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AL-KHALĪLĪ'S AUXILIARY TABLES FOR SOLVING PROBLEMS OF SPHERICAL ASTRONOMY

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Early in the development of Islamic astronomy there were tabulated sets of functions which had no direct astronomical significance, but were of such a nature that ordered applications of them would lead to the solution of spherical astronomical problems. These auxiliary tables reveal considerable mathematical ingenuity on the part of their original compilers. Thus far the tables of three important Islamic astronomers have been studied, namely, those of Ḥabash al-Ḥāsib[1] (*ca* 850); Abū Naṣr Manṣūr[2] (*ca* 1000); and Ibn Yūnus[3] (*ca* 1000). The purpose of the present paper is to discuss the auxiliary tables of al-Khalīlī[4] (*ca* 1375), an individual hitherto unrecognized in the literature as one of the greatest calculators of the Middle Ages. al-Khalīlī was associated with the Umayyad Mosque in Damascus,[5] and was a colleague of the distinguished astronomer Ibn al-Shāṭir. Very little is known yet about the activities of these Damascus astronomers.[6]

The tables of Ḥabash and Abū Naṣr are comparable to each other in that they are essentially single-argument tables with a total of a few hundred entries.[7] The tables of Ibn Yūnus are scattered throughout his *Zīj*, a lengthy compendium on astronomy, and are similar to those of his predecessor Ḥabash and his contemporary Abū Naṣr. However, Ibn Yūnus, with his *Very useful tables* for timekeeping,[8] was the forerunner of an important development in Islamic astronomy, namely, the compilation of extensive tables for timekeeping by the Sun and for regulating the astronomically defined times of Muslim prayer. His spherical astronomical tables, many of which are double-argument tables, contain over 30,000 entries, calculated with remarkable accuracy. These tables, computed for the latitude of Cairo-Fusṭāṭ, 30;0°, and for obliquity 23;35°, served as a model not only to a host of individuals who plagiarized them,[9] but also to certain astronomers who developed them. One of the latter was al-Khalīlī, who tabulated virtually all the functions tabulated by Ibn Yūnus, but for the latitude of Damascus, namely, 33;30°, and for obliquity 23;31°.[10] al-Khalīlī, however, had the ingenuity to devise a set of auxiliary functions which would work for all latitudes, at the cost of computing over 13,000 entries! This did not exhaust his ingenuity (or his patience): he also computed a table giving the *qibla*, that is, the direction of Mecca, for all latitudes and longitudes within the world of Islam.[11] This table contains over 3,000 entries, whose accuracy is quite remarkable, considering the nature of the function tabulated.

The auxiliary tables of al-Khalīlī and instructions on their use are to be found in at least three manuscript sources:[12]

MS. Paris Bibliothèque Nationale, ar. 2558, fols 61v–104r;[13]
MS. Princeton Yahuda 861,2;[14]
MS. Escorial ar. 931, fols 171r–211v.[15]

The first of these is a beautiful manuscript, carefully copied in an elegant hand, and dated 811 Hijra (=1408). It is a complete copy of al-Khalīlī's prayer-tables and tables for timekeeping.[16] The second manuscript is a carelessly copied version of the prayer-tables[17] of al-Wafā'ī (*ca* 1450), similar to those of Ibn Yūnus, but computed for each degree of latitude from 21° (Mecca) to 41° (Istanbul), and containing over 20,000 entries. al-Khalīlī's auxiliary tables have been bound in the middle of these tables. The third manuscript is a volume comprising several astronomical works, in which al-Khalīlī's auxiliary tables have been inserted at the end. These tables were copied in 933 Hijra (=1526).

al-Khalīlī tabulates three auxiliary functions, each to two sexagesimal digits. For the first two, the horizontal argument is terrestrial latitude, and I denote the functions by f_ϕ and g_ϕ, where ϕ is the local latitude. The vertical argument is defined in the instructions, and the functions are called *al-maḥfūẓ al-awwal* and *al-maḥfūẓ al-thānī* respectively, "first and second functions". The third function, which I denote by G, has a quantity called *jayb al-tartīb*, perhaps best rendered as "the auxiliary Sine", for horizontal argument, and the vertical argument is defined in the instructions. The *jayb al-tartīb* is an auxiliary function widely used in late Islamic astronomy, and its use will become apparent in what follows.

The functions are not explained in the instructions, but are in fact:

$$f_\phi(\theta) = \frac{R \operatorname{Sin} \theta}{\operatorname{Cos} \phi},$$

$$g_\phi(\theta) = \frac{\operatorname{Sin} \theta \operatorname{Tan} \phi}{R},$$

$$G(x,y) = \operatorname{arc\,Cos} \left\{ \frac{xR}{\operatorname{Cos} y} \right\},$$

where the trigonometric functions are to base $R = 60$.[18] These three functions are tabulated for the domains:[19]

$$\begin{aligned} \theta &= 1°, 2°, \ldots, 90° \\ \phi &= 1°, 2°, \ldots, 55°, \text{ and } 33;30° \text{ (Damascus)}, \\ x &= 1, 2, \ldots, 59, \\ y &= 0°, 1°, \ldots, n(x), \end{aligned}$$

where $n(x)$ is the largest integer such that

$$xR \leqslant \operatorname{Cos} n(x).$$

The values, written in standard Arabic sexagesimal notation,[20] are arranged in columns of thirty entries. On a given double opening one finds values of f_ϕ and g_ϕ for two consecutive values of ϕ, or values of G for four consecutive values of x. When the values of G are, in modern terms, non-real, no entry is given in the table. Columns which would be empty are omitted, and so, for example, the values of G for

$$x = 52, 53, \ldots, 59$$

fit on a double opening in MS. B.N. ar. 2558, fols 103v–104r. If the values of G were tabulated in a single table the resulting table would be trapezoidal in shape.

TABLE 1. Selected entries from the table of $f_\phi(\theta)$.
(Values are accurately computed, except where the error in the second digit is shown in square brackets.)

Argu-ment, θ	Latitude, ϕ 1°	2°	... 9°	... 19°	... 31°	... 49°	... 55°
1°	1; 3	1; 3	1; 4	1; 7[+1]	1;13	1;36	1;50
2°	2; 6	2; 6	2; 7	2;13	2;26[−1]	3;13[+1]	3;39
3°	3; 8	3; 8[−1]	3;10[−1]	3;20[+1]	3;39[−1]	4;49[+2]	5;29[+1]
⋮ 17°	17;32[−1]	17;33	17;45[−1]	18;33	20;29[+1]	26;45[+1]	30;36[+1]
⋮ 35°	34;25	34;26	34;50[+1]	36;23[−1]	40; 9	52;27	60; 0
⋮ 42°	40; 9	40;10	40;39	42;28	46;51[+1]	61;11[−1]	70; 0
⋮ 63°	53;28	53;30	54; 8	56;32	62;23[+1]	81;28[−1]	93;12
⋮ 78°	58;41[−1]	58;43	59;25	62; 4	68;28	89;26[−1]	102;19
⋮ 90°	60; 0[−1]	60; 2	60;44[−1]	63;28[+1]	70; 0	91;28[+1]	104;36

The entries in al-Khalīlī's tables were computed with remarkable precision. Most of the entries are accurate, and the error in the remainder is usually only ± 1 in the second digit, and occasionally ± 2. Errors in the table of G are slightly more frequent than in the tables of f_ϕ and g_ϕ, and larger isolated errors do occur. Tables 1–3 show random extracts from the three tables of al-Khalīlī, taken from MS. B.N. ar. 2558. The errors in the second digit are shown in square brackets, and are derived according to the convention

$$\text{error} = \text{text} - \text{recomputation}.$$

Table 4 shows a set of entries in the table of $G(x,y)$ for small values of x and large values of y, which are inaccurately computed. I am unable to explain how these errors might have arisen. In the Appendix to this paper I have indicated the kind of trigonometric and multiplication tables al-Khalīlī may have used to compile these auxiliary tables, which contain respectively 5,040, 5,040, and about 3,350 entries.

I shall now explain the method of using the tables, following al-Khalīlī's instructions. The rules which he outlines in words[21] are here expressed in modern mathematical notation and are numbered. I explain most of the rules by transforming the modern expression[22] for the function under discussion into a form which is equivalent to al-Khalīlī's rule, given the definitions of the functions f_ϕ, g_ϕ, and G. The arguments to be used in the tables are considered positive: in al-Khalīlī's introduction detailed instructions are given to cover all possible cases. Modern notation renders these superfluous in the commentary. The formulae on which his rules are based were well known in al-Khalīlī's time. Whilst some of them are not given in the *Zīj* of al-Battānī[23] (*ca* 900), all of them are contained in the *Ḥākimī Zīj* of Ibn Yūnus[24] (*ca* 1000).

TABLE 2. Selected entries from the table of $g_\phi(\theta)$.
(Values are accurately computed, except where the error in the second digit is shown in square brackets.)

Argument, θ	Latitude, ϕ 1°	2° ...	9° ...	18° ...	31° ...	49° ...	55°
1°	0; 1	0; 2	0;10	0;20	0;38	1;12	1;30
2°	0; 2	0; 4	0;20	0;41	1;16[+1]	2;25	3; 0[+1]
3° ⋮	0; 3	0; 6[−1]	0;30	1; 1	1;54[+1]	3;37	4;29
17° ⋮	0;18	0;36[−1]	2;46[−1]	5;42	10;32	20;10[−1]	25; 3
35° ⋮	0;36	1;12	5;27	11;11	20;41	39;36[+1]	49; 9
42° ⋮	0;42	1;25[+1]	6;21[−1]	13; 2[−1]	24; 6[−1]	46;10[−1]	57;20
63° ⋮	0;56	1;52	8;27[−1]	17;22	32; 7	61;31[+1]	76;21
78° ⋮	1: 2[+1]	2; 3	9;17[−1]	19; 4	35;16	67;30[−1]	83;48[−1]
90°	1: 3	2; 6	9;30	19;30	36; 3	69; 1	85;41

(i) To find the half diurnal arc, D, or half nocturnal arc, N, of a celestial body with declination δ,[25] given the terrestrial latitude ϕ, use:

$$\left.\begin{matrix} D \\ N \end{matrix}\right\} = G\,\{g_\phi(\delta), \delta\} \text{ for } \delta \lesseqgtr 0, \text{ respectively.} \tag{1}$$

In modern notation, for both $\delta \gtrless 0$:

$$D = 180^\circ - N = 90^\circ + \arcsin\{\tan\delta \tan\phi\},$$

so that:

$$\left.\begin{matrix} D \\ N \end{matrix}\right\} = \arccos\left\{\frac{\sin\delta\tan\phi}{\cos\delta}\right\} \text{ for } \delta \lesseqgtr 0, \text{ respectively.}$$

(ii) To find the hour-angle, t, given the solar altitude, h, for any declination and terrestrial latitude, use:[26]

$$t = G\,\{[f_\phi(h) - g_\phi(\delta)], \delta\}. \tag{2}$$

The modern formula is:

$$t = \arccos\left\{\frac{\sin h - \sin\delta\sin\phi}{\cos\delta\cos\phi}\right\}$$
$$= \arccos\left\{\frac{\dfrac{\sin h}{\cos\phi} - \sin\delta\tan\phi}{\cos\delta}\right\}.$$

(iii) To find the lengths of evening and morning twilight, r and s, assuming angles of solar depression 17° and 19° at nightfall and daybreak respectively,[27] use:

$$s = N - G\,\{[f_\phi(17^\circ) + g_\phi(\delta)], \delta\}, \tag{3}$$

$$r = N - G\,\{[f_\phi(19^\circ) + g_\phi(\delta)], \delta\}. \tag{4}$$

The length of twilight for solar longitude λ and declination δ is easily found[28] by calculating the time for the Sun to attain an altitude of 17° or 19° when the solar longitude is $\lambda + 180°$ and the declination is $-\delta$. The values of G in (3) and (4) define the corresponding hour-angles (*cf.* (2) above). Note also that:

$$N(\lambda) = D(\lambda + 180°).$$

(iv) To find the solar rising or ortive amplitude, ψ, use:

$$\psi = 90° - G\{g_{45}(\delta), \phi\}, \qquad (5)$$

or solve:

$$g_{45}(\psi) = f_{\phi}(\delta). \qquad (6)$$

In modern notation:

$$\psi = \arcsin\left\{\frac{\sin\delta}{\cos\phi}\right\},$$

so that:

$$\psi = 90° - \arccos\left\{\frac{\sin\delta\tan 45°}{\cos\phi}\right\},$$

and also:

$$\sin\psi\tan 45° = \frac{\sin\delta}{\cos\phi}.$$

Note that $g_{45}(\theta)$ is the Sine function.

TABLE 3. Selected entries from the table of $G(x,y)$.
(Values are accurately computed, except where the error in the second digit is shown in square brackets.)

Argument, y	Auxiliary Sine, x 1	2 ...	11 ...	20 ...	32 ...	46 ...	59
0°	89; 3°	88; 6[+1]	79;26	70;32	57;45[−1]	39;56[−1]	10;28[−1]
1°	89; 3	88; 6[+1]	79;26	70;32	57;45[−1]	39;56	10;26
2°	89; 3	88; 6[+1]	79;26	70;31	57;44[−1]	39;54	10;17
⋮ 13°	89; 1	88; 3[+1]	79; 9	70; 0	56;48[−1]	38; 6[−1]	—
⋮ 32°	88;52	87;45	77;31	66;[51][b]	51; 2	25;18	—
⋮ 47°	88;36	87;12	74;24	60;43[−1]	38;33	—	—
⋮ 68°	87;26[−1]	84;54	60;42	27;10[+1]	—	—	—
⋮ 74°	86;31[−1]	83; 2[−1]	48;18	—	—	—	—
⋮ 89°	57;15[a]	—	—	—	—	—	—

[a] *Cf.* Table 4.
[b] Text: 41. The restoration is confirmed by the accuracy of the adjacent entries.

(v) To find the altitude of a celestial body in the prime vertical, h_0, use:

$$h_0 = 90° - G\ \{g_{45}(\delta), \bar{\phi}\} \tag{7}$$

(MS. B.N. ar. 2558 has incorrectly:

$$h_0 = G\ \{g_{45}(\delta), \bar{\phi}\}),$$

where $\bar{\phi}$ is the complement of ϕ, or solve:

$$g_\phi(h_0) = f_\phi(\delta). \tag{8}$$

In modern notation:

$$h_0 = \text{arc sin} \left\{\frac{\sin \delta}{\sin \phi}\right\},$$

so that:

$$h_0 = 90° - \text{arc cos} \left\{\frac{\sin \delta \tan 45°}{\cos \bar{\phi}}\right\},$$

and also:

$$\sin h_0 \tan \phi = \frac{\sin \delta}{\cos \phi}.$$

(vi) To find the solar azimuth, a, measured from the meridian, use:

$$a = G\ \{[g_\phi(h) - f_\phi(\delta)], h\}. \tag{9}$$

The modern formula is:

$$a = \text{arc cos} \left\{\frac{\sin h \sin \phi - \sin \delta}{\cos h \cos \phi}\right\}$$

$$= \text{arc cos} \left\{\frac{\sin h \tan \phi - \dfrac{\sin \delta}{\cos \phi}}{\cos h}\right\}.$$

(vii) To find the solar declination from an observed pair of values (h,a), when the local latitude is known, first solve for x in:[29]

$$a = G\ \{x,h\}, \tag{10}$$

and then solve for δ in:

$$f_\phi(\delta) = g_\phi(h) - x. \tag{11}$$

This procedure is the reverse of (9) above.

(viii) To find both the solar declination and local latitude from two such pairs, subscripted 1 and 2, observed on the same day, first solve (10) for x_1 and x_2. Then define the quantity:

$$\Delta x = x_2 - x_1 \tag{12}$$

called the argument (*ḥiṣṣa*), and the quantity:

$$\Delta g = g_{45}(h_2) - g_{45}(h_1). \tag{13}$$

Next find the quantity ξ, called the correction (*ta'dīl*), defined by:

$$g_{45}(\xi) = \Delta g. \tag{14}$$

TABLE 4. Selected entries in the table of $G(x,y)$ containing anomalous errors.

Argument, y	Auxiliary Sine, x						
	1	2	3	4	5	6	7
						⋮	⋮
81°					⋮	44; 3[b]	41;45
82				⋮	53;12	34;[55][b,c]	33; 2
83			⋮	56;52	46;53	16;46[b]	16;55
84		⋮	61;24	50;20	37; 6	—	—
85	⋮	67;32	55; 1	40; 9	16;10	—	—
86	76;10	61;26	44;10	17; 1	—	—	—
87	71;23	50;20	16;46	—	—	—	—
88	61;34	17;45	—	—	—	—	—
89	57;15[a]	—	—	—	—	—	—

[a] Whilst it is tempting to restore 57 to 17, it is also possible that al-Khalīlī computed R/Cos 89° as 57;15 (accurately, 57;18) and simply forgot to take the arc Cosine. MS Escorial ar. 931, fol. 196v also has 57;15.
[b] The entry for y = 60° has been omitted from this column and each of the entries is one space too high.
[c] Text: 15.

Corresponding recomputed values

	1	2	3	4	5	6	7
							⋮
81°						⋮	41;46
82				⋮	53;13	44; 4	33; 2
83			⋮	56;50	46;52	34;52	16;48
84		⋮	61;25	50;22	37; 8	16;56	—
85	⋮	67;31	55; 0	40; 6	17; 2	—	—
86	76;11	61;27	44;13	17; 7	—	—	—
87	71;26	50;26	17;11	—	—	—	—
88	61;28	17;14	—	—	—	—	—
89	17;15	—	—	—	—	—	—

Then solve for ϕ in:

$$g_{\phi}(\xi) = \Delta x. \tag{15}$$

Finally, the declination is to be found from (11) above. The procedures are easily verified by projecting the celestial sphere onto the meridian plane.[30]

(ix) To find the declination of a star, Δ, from its ecliptic coordinates (λ,β), first find λ'', the longitude difference between the star and the nearest solstice, and then solve for x in:

$$\lambda'' = G\,\{x,\beta\}. \tag{16}$$

The declination is then found by solving:[31]

$$f_{37}(\Delta) = \tfrac{1}{2}\,\{2g_{49}(\beta) + x\}. \tag{17}$$

To explain this procedure, let ϵ denote the obliquity of the ecliptic. The modern formula is then:

$$\Delta = \text{arc sin}\,\{\sin\beta\cos\epsilon + \cos\beta\sin\lambda\sin\epsilon\},$$

so that:

$$\sin \Delta = \sin \beta \cos \epsilon + \cos \beta \sin \lambda \sin \epsilon,$$

and:

$$\frac{\sin \Delta}{\sin \epsilon} = \frac{\sin \beta \cos \epsilon}{\sin \epsilon} + \cos \beta \sin \lambda.$$

In order to adapt this formula for use with al-Khalīlī's tables, we write:

$$\frac{\sin \Delta}{\cos \phi_1} = \tfrac{1}{2} \{2 \sin \beta \tan \phi_2 + \cos \beta \sin \lambda\}.$$

It follows that the values ϕ_1 and ϕ_2 are defined by:

$$\tfrac{1}{2} \cos \phi_1 = \sin \epsilon \text{ and } 2 \tan \phi_2 = \cot \epsilon.$$

Since al-Khalīlī, following Ibn al-Shāṭir, used $\epsilon = 23;31°$, the accurate values of ϕ_1 and ϕ_2 are:

$$\phi_1 = 37;4° \text{ and } \phi_2 = 48;58°.$$

al-Khalīlī used the approximations:

$$\phi_1 = 37° \text{ and } \phi_2 = 49°.$$

(x) To find the normed right ascension α' of a star,[32] use:

$$\alpha'' = G \{[2f_{37}(\beta) - 2g_{49}(\Delta)], \Delta\}, \tag{18}$$

where α'' is the normed right ascension measured from $\alpha' = 0°$ or $\alpha' = 180°$.

The modern formula for right ascension, α, may be written:

$$\alpha = \arcsin \left\{ \frac{\sin \Delta \cos \epsilon - \sin \beta}{\cos \Delta \sin \epsilon} \right\}$$

$$= \arcsin \left\{ \frac{\sin \Delta \cot \epsilon - \dfrac{\sin \beta}{\sin \epsilon}}{\cos \Delta} \right\},$$

so that:

$$\alpha' = \arccos \left\{ \frac{- \left[2 \sin \Delta \tan \phi_2 - \dfrac{2 \sin \beta}{\cos \phi_1} \right]}{\cos \Delta} \right\}$$

$$= \arccos \left\{ \frac{\dfrac{2 \sin \beta}{\cos \phi_1} - 2 \sin \Delta \tan \phi_2}{\cos \Delta} \right\},$$

where ϕ_1 and ϕ_2 are as defined in (ix) above.

In modern terms, it may be stated that al-Khalīlī's tables are auxiliary tables for solving problems of spherical astronomy which involve the cosine rule for spherical triangles. The formulae which define the functions treated by al-Khalīlī were well known to Islamic astronomers already in 'Abbāsid times, but no Islamic treatise dealing with the spherical cosine rule in its generality is known from the medieval period. The formulae for the hour-angle and solar azimuth in terms of solar altitude, and for the declination and right ascension of a star in terms of its ecliptic coordinates can however be derived without difficulty by projection methods.

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REFERENCES

1. R. Irani's thesis (Irani [1]) remains to be published. On Ḥabash, see Kennedy [1], 126–7 (nos. 15 and 16).
2. See Jensen [1].
3. See King [1], 29–33.
4. On Shams al-Dīn Abū 'Abd Allāh Muḥammad b. Muḥammad al-Khalīlī, see Suter [1], no. 418.
5. Suter (*cf.* ref. 4) states that al-Khalīlī was associated with the Yalbaghā Mosque outside Damascus. The title folio of MS B.N. ar. 2558 states that he was a timekeeper (*muwaqqit*) at the Umayyad Mosque.
6. A. Sayili has already pointed to the need for further research in this area. *Cf.* Sayili [1], 245. On Ibn al-Shāṭir and his work, see Kennedy [1], 125 (no. 11); Wiedemann [1], ii, 729–38; and the recent studies listed by N. Swerdlow in *Journal for the history of astronomy*, iii (1972), 48, ref. 3.
7. C. Jensen has published a few entries from Abū Naṣr's table. The following table shows the corresponding recomputed values, to be compared with those in Jensen [1], 4.

I	II	III	IV	V	VI	VII
1°	2°	;24,16,10,41	;0,27,47,16	;1,54,54,58	;2, 5,38,17	;54,52,50,12
2	4	;24,16,50,37	;0,55,35,33	;3,49,45,39	;4,11, 7,24	;54,54, 0,38
3	6	;24,17,57,15	;1,23,25,52	;5,44,27,44	;6,16,18, 9	;54,55,57,36
4	8	;24,19,30,41	;1,51,19,15	;7,38,56,50	;8,21, 1,23	;54,58,40,27
44	88	;33,44, 1, 2	;25,37,20,18	;59,57,22,43	;59,57,48,25	;59,59,38,29
45	90	;34,19, 1,59	;26,31,57,38	;60, 0, 0, 0	;60, 0, 0, 0	;60, 0, 0, 0

It is clear that in numerous entries the third of the four digits is in error, and the fourth is meaningless. Furthermore, had Abū Naṣr used the value 23;35° for the obliquity, which was generally accepted in his time, rather than Ptolemy's value 23;51,20°, the entries in all but column VI (which are for the sine function) would generally differ even in the second digit from those given in his table.

It is perhaps worth remarking that al-Nayrīzī (*ca* 900) also tabulated some auxiliary functions. Two tables attributed to him, based on the *Khaṇḍa Khādyaka* value $R = 2,30$, are located in MS Berlin 5750, fols 83r–85r, of Ḥabash's *Zīj*. *Cf.* King [1], 29.

8. *Cf.* King [2] for a detailed analysis.
9. *Cf.* King [1], 43 and [2].
10. An analysis is in preparation.
11. This table is analyzed in detail in King [3]. It may be that al-Khalīlī computed the entries in the *qibla* table using his auxiliary tables.
12. I have not inspected MSS Berlin 5754–56, British Museum 977, 31, and Oxford I. 961, 1039,2, listed by Suter. *Cf.* ref. 4 above.

13. *Cf.* de Slane [1], 460–1.
14. I am indebted to Prof. R. Mach of Princeton University for kindly allowing me to consult his unpublished catalogue of the Yahuda Collection.
15. *Cf.* Renaud [1], 42–3. In this MS the tables are attributed to Sharaf al-Dīn Abū 'Abd Allāh al-Khalīlī, who lived about the year 1400. *Cf.* Suter [1], no. 427. Probably Sharaf al-Dīn should be restored to Shams al-Dīn.
16. *Cf.* ref. 10 above.
17. An analysis is in preparation. On al-Wafā'ī see Suter [1], no. 437, and King [1], 43 and 49.
18. The capital letter notation, now standard, is used to denote medieval trigonometric functions. *Cf.* Kennedy [1], 139b–140a.
19. In the Escorial MS, the maximum value of ϕ is 50°. As stated by Abū Naṣr, beyond latitude 45° there were few people who knew about spherical astronomy or even thought about it. *Cf.* Jensen [1], 5.
20. On which, see Irani [2].
21. *Cf.* refs 26, 29, and 31 below.
22. For some of the modern formulae, consult Smart [1], 25–52. For the derivation of these formulae by medieval methods, see, for example, King [1], 90–316.
23. On which, see Kennedy [1], 132–3 (no. 55), and the edition of C. Nallino (Milan-Rome, 1899–1907). al-Battānī does not, for example, discuss the calculation of the duration of twilight.
24. With the reservation that Ibn Yūnus discusses the calculation of time since sunrise rather than the hour-angle. *Cf.* King [1], 147–52.
25. Generally, I use δ for solar declination (*mayl*) and Δ for the declination of a star (*bu'd*). For most of the problems under discussion, al-Khalīlī notes the validity of his procedures for both the Sun and the fixed stars.
26. al-Khalīlī's instructions (MS B.N. ar. 2558, fol. 62r, ll. 1–7) read:

    Section on the hour-angle. Find the value of the first function for the altitude (as argument) on the page corresponding to the latitude, and find the value of the second function (for the declination as argument) as we described above for the half diurnal arc. Then, if the Sun is in the southern signs, add the two functions; otherwise, take the difference between them. The result will be the auxiliary Sine. Enter it in the table with the auxiliary Sine (as horizontal argument) and enter the declination as (vertical) argument: you will find the hour-angle. However, if the Sun is in the northern signs and the value of the first function is less than the value of the second, subtract the result from 180°: the remainder will be the hour-angle. If the two functions are equal, the hour-angle will be 90°.
27. On Islamic values for these parameters, consult Wiedemann [1], ii, 769 and 777, and King [2].
28. See, in particular, Smart [1], 51–52.
29. al-Khalīlī's instructions for these operations (MS B.N. ar. 2558, fol. 62v, ll. 15–23) read as follows:

    If either one of the meridian or the prime vertical is known, then the azimuth of an observed solar altitude can be found. Enter the altitude and azimuth in the auxiliary Sine tables in the following way. Look for a value in the table equal to the azimuth, which has the altitude as (vertical) argument. The auxiliary Sine is then found (as the horizontal argument) at the head of the appropriate table. Next find the value of the second function for the altitude on the appropriate page for the latitude, and subtract the value from the auxiliary Sine if the Sun is in the southern signs. If it is in the northern signs and the azimuth is northern, add the two values. Otherwise take the difference between them. The result is a value of the first function. Enter the value on the appropriate latitude page and find the corresponding vertical argument. This will be the declination.
30. *Cf.* King [1], 196, for Ibn Yūnus's treatment of this problem.
31. al-Khalīlī describes this method as follows (MS B.N. ar. 2558, fols. 63r, l. 27–63v, l. 4):

    . . . Enter the latitude of the star as argument for the second function for latitude 49°. Then double the result and add it to the auxiliary Sine. . . . Enter the result in the table of the first function for latitude 37° and find the corresponding vertical argument. It will be the declination of the star. . . .

    He does not explain the reason for using the tables for latitudes 49° and 37°.
32. The normed right ascension is defined by:

$$\alpha'(\lambda) = \alpha(\lambda) + 90°.$$

This function is tabulated in the *Handy tables*, and similar tables are common in Islamic *zījes*. It is useful for finding the ascendant, H, if the longitude of upper midheaven, M, is known, since:

$$\alpha\phi(\lambda_H) = \alpha(\lambda_M) + 90° = \alpha'(\lambda_M),$$

where $\alpha\phi$ denotes oblique ascension for latitude ϕ.

APPENDIX

There is no explicit indication in the texts consulted of the method used by al-Khalīlī to compile the tables. He clearly did not compute all of the entries directly, and the error patterns confirm that he used some kind of interpolation system. It may be that he computed the values of:

$$\frac{\text{Sec}\ \theta}{R} = \frac{R}{\text{Cos}\ \theta} \qquad (\theta = 1°, 2°, \ldots, 89°)$$

since these are required to compute:

$$G(1,y) = \text{arc Cos}\left\{\frac{R}{\text{Cos}\ y}\right\}.$$

It is reasonable to assume that he used the Secant function to compile the tables, rather than continually dividing by the Cosine in the tabulation of f_ϕ and G. No independent table of the Secant is contained in any known Islamic source, and the only Islamic table of the Cosecant function is in MS Berlin 5750, fols. 85r–87v, which may be due to Ḥabash (*cf.* Kennedy [1], 151b).

It is also possible that al-Khalīlī computed a table of arc Cos (z). No table of this kind is attested yet in the sources, and the only Islamic tables of arc Sin (z) known to me are contained in the Yemeni *Mukhtār Zīj* (MS British Museum 768 (Or. 3624), fol. 169v), and in one manuscript of Ibn Yūnus's *Very useful tables* (MS Chester Beatty 3673, fol. 8r). The function is computed to two sexagesimal digits for each 0;30 and 1;0 of argument, respectively, in the two tables. Tables of the Sine and Tangent function, accurately computed to, say, three sexagesimal digits, for each degree of argument were standard in the time of al-Khalīlī. He may have used the trigonometric tables of Ibn al-Shāṭir, in which the Sine is given to four sexagesimal digits for each degree of argument and the Tangent to three digits for each half degree (*cf.* Kennedy [1], 162b–163a). Alternatively, he might have used Sine tables like those of Ibn Yūnus, computed to five sexagesimal digits for each minute of argument (*cf.* King [1], 9 and 85–89), to take the arc Cosine in the compilation of the table of G.

Finally, it is certain that al-Khalīlī had at his disposal a multiplication table containing the sexagesimal products:

$$m \times n \qquad (m, n = 1, 2, \ldots, 60)$$

such as, for example,

$$34 \times 13 = 7{,}22 \text{ and } 41 \times 23 = 15{,}43.$$

Multiplication tables are not generally contained in Islamic *zījes* or tables for timekeeping. Indeed' only one complete Islamic multiplication table has been noted in the literature. P. Luckey has described a table in a work by the Egyptian scholar Sibṭ al-Māridīnī (d. *ca* 1500, *cf.* Suter [1], no. 445, and Luckey [1], 68, and 39, n. 67). The multiplication table referred to by Kūshyār b. Labbān (*fl. ca* 1000, *cf.* Suter [1], no. 192, and Levey [1], 98 and 36–38) has been omitted from the manuscript published by M. Levey. Likewise, the Samarqand mathematician and astronomer al-Kāshī (*fl. ca* 1430, *cf.* Luckey [1], esp. 47) in his treatise on arithmetic describes but does not reproduce the multiplication table which he used for his extensive calculations. R. Irani has discussed an extract from such a table contained in MS Princeton Yahuda 373 (*cf.* Irani [3]). The following examples of complete Islamic sexagesimal multiplication tables have recently come to my attention, and there is every reason to suppose that numerous others survive in the vast manuscript sources. These tables were the standard computational aids of Islamic astronomers and mathematicians.

(*a*) MS B.N. ar. 2531 is a work by the prolific Egyptian scholar Ibn al-Majdī (d. 1447, *cf.* Suter [1], no. 432), dealing with the planetary tables of Ibn Yūnus and Ibn al-Shāṭir. Fols. 115v–127r contain a table of sexagesimal products:

$$m \times n \qquad \begin{cases} m = 1, 2, \ldots, 120 \\ n = 1, 2, \ldots, 60 \end{cases}.$$

(*b*) MS B.N. ar. 2520 is an anonymous *Zīj* comprising tables originally due to Ibn Yūnus and Ibn al-Shāṭir. Fols 167v–173r contain a table of products

$$m \times n \qquad (m, n = 1, 2, \ldots, 60).$$

At the head of each of the sixty columns for m, the value of $\sqrt{m}$ is given to three sexagesimal digits.

(*c*) MS Princeton Yahuda 4072, fols 23r–29r, and

(*d*) MS Aleppo Aḥmadīya 1310. Both these sources contain tables similar to those in (*b*) above. In the second table m^2 is given at the head of each of the sixty columns for m.

(*e*) MSS Munich 865 (313) and 866 (318). I have not inspected these sources, but the description of them in the catalogue (Aumer [1], 380) indicates that they contain a multiplication table.

These tables are referred to in the sources as either *al-jadwal al-sittīnī* or *jadwal al-nisba al-sittīnīya*, "sexagesimal table".

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